Calculus I Lab 1
Limits

In this Lab you will study limits both numerically and graphically. You will first create tables to make a guess for the limit and then you will graph the function to verify your guess. Finally you will use *Mathematica* to compute the limit exactly. Make your work look nice by labeling each problem, using text cells to answer questions in complete sentences, and deleting any blue error messages.

Problem 1

Let f(x) = $\dfrac{\sqrt{x-1}-\cos(\pi x)}{x-2}$.

a. Define f(x) in *Mathematica*. Be very careful to use **( )** and **[ ]** where needed. Check that you've defined f(x) correctly.

b. Make a guess accurate to three decimal places for $\lim_{x\to 2} f(x)$ by creating tables for the values of f(x) for x near 2. Turn in only the table from which you make your guess.

c. Verify your guess in b. by plotting the graph of y = f(x) near x=2. Label the axes appropriately.

d. Let *Mathematica* compute the limit for you by entering the following exactly:

```
Limit[f[x], x → 2]
```

e. How good was your guess?

f. Clear f and x.

Problem 2

Let g(x) = $\begin{cases} x+2 & \text{if } x < -3 \\ 7-x^2 & \text{if } x \geq -3 \end{cases}$.

a. Define g(x) in *Mathematica*. Refer to **Practice #1** on how to define a piecewise function. Check that you've entered g(x) correctly by computing g(-3) and g(-3.1).

b. Make a guess, if you can, accurate to three decimal places for $\lim_{x\to -3} g(x)$ by creating tables for the values of g(x) for x near -3. Turn in only the table from which you made your guess. If you can't seem to make a guess, explain why.

c. Verify your answer in b. by plotting the graph of y = g(x). Be sure to plot the graph without the vertical line. Refer to **Practice #2** on how to do this.

d. Use the theory from class to explain what is going on with this limit.

e. Clear g and x.

Problem 3
Let h(x)=$\frac{1-\cos x}{x^3}$.

a. Define h(x) in *Mathematica.* Check that you've defined h(x) correctly.

b. Compute $\lim_{x\to 0}$ h(x) in *Mathematica*. Use a table or graph to show that *Mathematica* is wrong. What is the correct answer to this limit?

Calculus I Lab 2
Tangents and Secants

In this Lab you will look at secant and tangent lines both numerically and graphically and try to make a connection between continuity and tangent lines. Be sure to make your work look nice. PLEASE, PLEASE, PLEASE be careful when creating tables. Make sure that you're creating the table for what's being asked of you. Also, clear f, x, and a before moving on to the next problem.

Problem 1
Let f(x) = $-x^4 + 2x^3 - 3x + 10$.

a. Define f(x) in *Mathematica*. Since we will be looking at slopes of many secant lines, define the following as written

```
msecant[x_, a_] := (f[x] - f[a]) / (x - a)
```

This **msecant** will compute the slope of the secant line to f from x to a. Technically this is a function of two variables. We do this because it's easier to change the **a** than it is to retype the quotient for each **a**.

b. Plot the graph of y = f(x) on the interval [1,3]. Is f continuous? We want to examine the slope of the tangent line to f at x=2. Is it positive or negative? Explain. Now "zoom in" on the graph around x=2. That is, plot the graph on smaller and smaller intervals containing 2. Try [1.5,2.5], [1.9,2.1], and [1.99,2.01]. What do you notice about the graph as you zoom in? Turn in only the first and last graphs.

c. Let a=2 in *Mathematica*. That is, just enter a=2. Create tables of values for the slopes of the secant lines to f through x=a. Make a guess for the slope of the tangent line at x=a good to two decimals. Turn in only your last table.

d. Use *Mathematica* to compute the exact slope of the tangent line to f at a=2. How good was your guess in c.?

e. Find the equation of the tangent line to f at x=a using pencil and paper. Define it in *Mathematica* as the function **tangentline[x]**.

f. Plot the graphs of f and its tangent line on the same intervals as you did in b. Use different styles to distinguish the graphs(refer to the Introduction). What do you notice?

Problem 2

Let f(x) = $\begin{cases} 2x + 8 & \text{if } x < -2 \\ 2 - x & \text{if } x \geq -2 \end{cases}$.

a. Define f(x) in *Mathematica* and plot its graph. Is f continuous at x=-2? Do you think that f has a tangent line at x=-2?

b. Let a=-2 in *Mathematica*. Confirm your answer from a. by making a table of values for the slopes of the secant lines to f through x=-2. Get close enough to x=-2 to make a guess for the slope of the tangent line. If you can't seem to make a guess, explain what you think is going on. Now do you think that f has a tangent line at x=-2? Turn in only the last table.

c. Based on your observations from this and **Problem 1**, complete the following sentence:
If f(x) is continuous at x=a, then f(x) ____ a tangent line at x=a.

Problem 3

Let f(x) = $\begin{cases} x - 2 & \text{if } x < 3 \\ 1 - x & \text{if } x \geq 3 \end{cases}$.

a. Define f(x) in *Mathematica* and plot its graph. Is f continuous at x=3? Do you think that f has a tangent line at x=3?

b. Let a=3 in *Mathematica*. Confirm your answer a. by making a table of values for the slopes of the secant lines to f through x=3. Get close enough to x=3 to make a guess for the slope of the tangent line. If you can't seem to make a guess, explain what you think is going on. Now do you think that f has a tangent line at x=3? Turn in only the last table.

c. Based on your observation, complete the following sentence:
If f(x) is not continuous at x=a, then f(x) __________________________ a tangent line at x=a.

Problem 4

Let f(x) = $(x-1)^{2/3}$.

a. Define f(x) in *Mathematica* by entering the following:

```
f[x_] := ((x - 1) ^2) ^ (1 / 3)
```

b. Plot the graph of f(x) on the interval [0,2]. Is f continuous? Now zoom in around x=1. Do you think that f has a tangent line at x=1? Turn in only the first graph.

c. Let a=1 in *Mathematica*. Make a table for the slopes of the secant lines to f through x=1. Get close enough that you can make a guess for the slope of the tangent line at x=1. What is your guess? What type
of line is the tangent line? What is its equation?

Calculus I Lab 3
Horizontal Tangents

In this Lab you will use *Mathematica* to solve the Horizontal Tangent Line problem. What's that? Did someone say "tangent line?" You must need derivatives. What does horizontal mean? Yes, the slope is 0. This must mean that this Lab is really about finding where the derivative is 0. First, you'll learn how to compute derivatives in *Mathematica*. Second you'll learn how to solve equations. Third, you'll put these together to solve the main problem. Answer all questions in a notebook file and use text cells for text. As always, make your work look nice.

Derivatives

The syntax for computing derivatives in *Mathematica* is the easiest of all. To find the derivative of f with respect to x, you enter **D[f[x],x]** or **f'[x]**. Here is an example: Find f '(x) for f(x) = $x^2 \sin 2x$. The first way does it without defining f(x) and second way does. If you know you're going to use f(x) again, do it the second way.

```
D[x^2 * Sin[2 x], x]
```

```
2 x^2 Cos[2 x] + 2 x Sin[2 x]
```

```
f[x_] := x^2 * Sin[2 x]
f'[x]
```

```
2 x^2 Cos[2 x] + 2 x Sin[2 x]
```

In class you talked about "differentiating a function with respect to a variable." You will see more of what we mean by this now. Suppose we want to differentiate g(t) = sin t. Watch what happens:

```
D[Sin[t], x]
```

```
0
```

You certainly know that the derivative of sin t is cos t. So how does *Mathematica* get 0? We told it to differentiate a function of t WITH RESPECT TO x and, by default, *Mathematica* takes x to be the variable and t to be a constant.

WHEN YOU WANT THE DERIVATIVE OF A FUNCTION, BE SURE THAT THE VARIABLE WITH RESPECT TO WHICH YOU'RE DIFFERENTIATING IS THE VARIABLE OF YOUR FUNCTION.

You should enter:

```
D[Sin[t], t]
```

```
Cos[t]
```

Problem 1

Find the derivatives of each of the following functions. Be sure that you've entered the function correctly. **Simplify** your answers to make them look nicer.

a. f(x) = $\frac{x^3-3x^2-x}{2x^2-x+2}$ b. $g(t)$ = sin2t tan3t c. h(r) = $\sqrt{r^3-3r^2-r\cos r}$

Solving Equations

Here's how you solve the equation $x^2 - 5 = 0$:

```
Solve[x^2 - 5 == 0, x]
```

$\{\{x \to -\sqrt{5}\}, \{x \to \sqrt{5}\}\}$

Note that the equation is written **with two equal signs**. The **x** means to solve for x. The output means that x =$\sqrt{5}$ or x = -$\sqrt{5}$. *Mathematica* does not assign x to be $\pm\sqrt{5}$.

There are equations for which **Solve** does not work. That is, *Mathematica* cannot solve them exactly. We have to find decimal approximations using **NSolve**.

```
Solve[x^5 - x^4 + 4 x^3 - x^2 + 2 x - 1 == 0, x]
```

```
{{x → Root[-1 + 2 #1 - #1^2 + 4 #1^3 - #1^4 + #1^5 &, 1]},
 {x → Root[-1 + 2 #1 - #1^2 + 4 #1^3 - #1^4 + #1^5 &, 2]}, {x → Root[-1 + 2 #1 - #1^2 + 4 #1^3 - #1^4 + #1^5 &, 3]},
 {x → Root[-1 + 2 #1 - #1^2 + 4 #1^3 - #1^4 + #1^5 &, 4]}, {x → Root[-1 + 2 #1 - #1^2 + 4 #1^3 - #1^4 + #1^5 &, 5]}}
```

```
NSolve[x^5 - x^4 + 4 x^3 - x^2 + 2 x - 1 == 0, x]
```

```
{{x → -0.184108 - 0.794563 i}, {x → -0.184108 + 0.794563 i},
 {x → 0.438127}, {x → 0.465044 - 1.79299 i}, {x → 0.465044 + 1.79299 i}}
```

Unfortunately, there are equations for which neither **Solve** nor **NSolve** will work. Here is an example. Let's try to find a nonzero solution to sinx = x/2.

```
Solve[Sin[x] == x / 2, x]
```

```
Solve::tdep : The equations appear to involve the variables to be solved for in an essentially non-algebraic way.
```

Solve$\left[\text{Sin}[x] == \frac{x}{2}, x\right]$

```
NSolve[Sin[x] == x / 2, x]
```

```
Solve::tdep : The equations appear to involve the variables to be solved for in an essentially non-algebraic way.
```

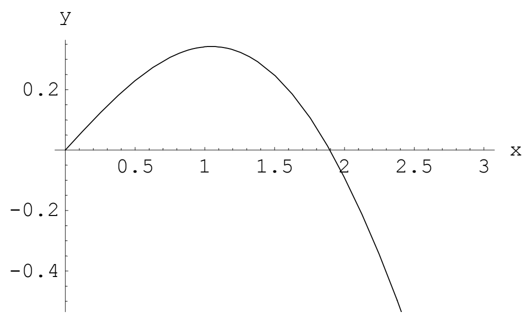
NSolve$\left[\text{Sin}[x] == \frac{x}{2}, x\right]$

We need the command **FindRoot**, which requires a decent guess for the answer. How do you make a guess for the answer? First, it's probably easier to make the equation equal to 0. This is why **FindRoot** is called

FindRoot: A number r is called a *root* of a function provided f(r)=0. That is, r is a solution to the equation f(x)=0. A root is also then an x-intercept for the graph of the function. So graph the function and look for x-intercepts. You can see that 1.9 is a good guess. Almost any number can be a guess, but the closer the guess to the actual root, the better.

```
f[x_] := Sin[x] - x / 2

Plot[f[x], {x, 0, 3}, AxesLabel → {"x", "y"}];
```




```
FindRoot[f[x] == 0, {x, 1.9}]

{x → 1.89549}
```

Problem 2

Solve each of the following equations. If **Solve** doesn't work, use **NSolve**. If **NSolve** doesn't work, use **FindRoot**. In any case, be sure to find ALL solutions to each equation and explain how you know you have done so. Also, ignore any complex solutions(those with i). Remember that the tangent function has vertical asymptotes. In particular, those for tan2x are x=π/4 and x=3π/4. You may have to plot your function on several intervals to find your guesses.

a. $x^6 - x^5 + 4x^3 - 5x = 8$ b. sin2x + cos2x = x c. tan2x = $\sqrt{x}$ on [0,π]

Problem 3

Now put together what you've done in **Problems 1** and **2** to find where the following functions have horizontal tangent lines. Don't let *Mathematica* fool you with its graphs.You may have to zoom in or out to get a better look, especially in c. As in **Problem 2**, be sure that you've found all the solutions.

a. f(x) = $x^7 - 2x^6 - x^5 + 4x^3 - 5x + 2$ b. g(x) = sin2x + x/4 on [0,2π]

c. h(x) =$x^4 - 200x^3 - 333x^2 + 1000x$

Calculus I Lab 4
Implicit Differentiation

Plotting Implicit Graphs

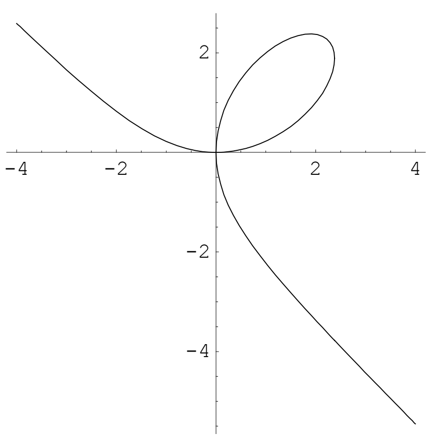
Open a new notebook file and, before doing anything else, enter the command below. When you re-open this notebook later, the first thing you must do is re-enter this command. If you don't, you may not be able to plot the graphs.

```
<< Graphics`ImplicitPlot`
```

In this Lab you will examine the tangent lines to the graphs of implicit functions. In particular, you are going to look at when these tangents are horizontal or vertical. This Lab will require you to use both *Mathematica* and your brain. That is, some work will be done on the computer and some will be done with pencil and paper. You will learn to make substitutions in *Mathematica*.

After you have entered the above command, you can start graphing implicit functions. In *Mathematica* lingo, what you've done is "loaded the ImplicitPlot package." *Mathematica* has lots of these packages that do more than the usual *Mathematica* does. (You can find out more about these "Standard Packages" in the *Help Browser* under *Add-ons*.) For the most part you can get away with telling *Mathematica* the x values on which to graph. For example, graph $x^3 + y^3 = \frac{9\,xy}{2}$ for -4≤x≤4. Note the two ='s in the equation and the space between the x and y. The variable **ip** is used since this graph will be needed later.

```
ip = ImplicitPlot[x^3 + y^3 == 9 x y / 2, {x, -4, 4}];
```



The graph is nice and smooth, so it should have tangent lines. And if it has tangent lines, then it has a derivative. First you have to tell *Mathematica* that y is a function of x, then differentiate both sides with respect to x. If you need to plot another implicit graph after this, you will have to clear **y**. If you don't, you won't be able to do the graph. Note again the two ='s and the space between the x and y.

```
y := f[x]
D[x^3 + y^3 == 9 x y / 2, x]
```

$$3 x^2 + 3 f[x]^2 f'[x] == \frac{9 f[x]}{2} + \frac{9}{2} x f'[x]$$

Now f '[x] is what you're looking for. So you have to solve for f '[x] and then use the command **FullSimplify**, which does more than **Simplify**, to simplify the answer.

```
Solve[%, f'[x]];
FullSimplify[%]
```

$$\left\{\left\{f'[x] \to \frac{2 x^2 - 3 f[x]}{3 x - 2 f[x]^2}\right\}\right\}$$

Read this as $\frac{dy}{dx} = \frac{2x^2 - 3y}{3x - 2y^2}$. Now you need *Mathematica* to evaluate this at various points on the graph. Recall that you had defined a function with two variables before: it was **msecant[x,a]** from **Lab 2**. Call the derivative **fprime** and use it to find the slope of the tangent line at the point (1,2). Then find the equation of the tangent line and plot it along with the original graph.

```
fprime[x_, y_] := (2 x^2 - 3 y) / (3 x - 2 y^2)
```

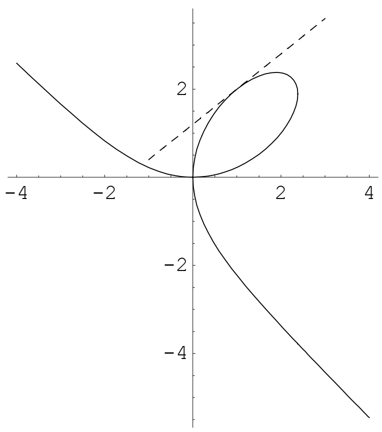
```
fprime[1, 2]
```

$$\frac{4}{5}$$

```
line1[x_] := 4 / 5 (x - 1) + 2
```

```
lineplot = Plot[line1[x], {x, -1, 3}, PlotStyle → Dashing[{.02}], DisplayFunction → Identity];
```

```
Show[ip, lineplot];
```



Problem 1

For each of the following equations,

Use **ImplicitPlot** to plot the graph on the given interval,

Use *Mathematica* to find $\frac{dy}{dx}$,

Define **fprime** as above and use it to find the slope of the tangent line at the given point, and

Plot the graph with its tangent line.

After completing a., be sure to **Clear x**, **y**, and **fprime** so you, and *Mathematica,* don't get confused.

a. $x^2 + y^3 = 2$ for $-4 \le x \le 4$, (-1,1)

b. $x^2 + y^2 - 2y = 2\sqrt{x^2 + y^2}$ for $-3 \le x \le 3$, (2,0)

Special Tangents

Now you will examine these implicit plots more closely. In particular, you will find where they have horizontal or vertical tangent lines. Take a look at our first graph. It looks like there are horizontal tangent lines at x=0, 1.8 and vertical tangents at x=0, 2.2. That x=0 thing is weird, so don't bother with it. How can you find these points with better accuracy? You know what the derivative is: $\frac{dy}{dx} = \frac{2x^2 - 3y}{3x - 2y^2}$. To have a horizontal tangent line, the derivative has to be 0. So the top = 0 and bottom ≠ 0. You have $2x^2 - 3y = 0$. Solve for y:(This can also be done by hand). If you do it in *Mathematica*, clear y first. Do not clear y between defining it as a function of x and computing $\frac{dy}{dx}$.

```
Clear[y]

Solve[2 x^2 - 3 y == 0, y]
```

$\{\{y \to \frac{2x^2}{3}\}\}$

Now plug this back into the original equation and solve for x. Mathematica can do this for you if you enter the following:

```
x^3 + y^3 == 9 x y / 2 /. y → 2 x^2 / 3
```

$x^3 + \frac{8x^6}{27} == 3x^3$

The **/.y→2x^2/3** means to substitute $\frac{2x^2}{3}$ for y. Now since the last output is a *Mathematica* equation, we can use the **%** to solve for x. Just use **NSolve** here since the **Solve** gives funny complex answers. We see that our x-value is about 1.89, rounded to two decimals. Now we need the y value. So plug in this x into the original equation and solve for y.

```
NSolve[%, x]
```

{{x → -0.944941 - 1.63669 ⅈ}, {x → -0.944941 + 1.63669 ⅈ}, {x → 0.}, {x → 1.88988}}

```
x^3 + y^3 == 9 x y / 2 /. x → 1.89
```

$6.75127 + y^3 == 8.505\, y$

```
NSolve[%, y]
```

```
{{y → -3.25277}, {y → 0.871673}, {y → 2.3811}}
```

Does it make any sense that there are three y values for this one x? Yes, since the vertical line x=1.89 intersects the graph three times. So does this mean that we have horizontal tangents at all three points? How can we check? Use the derivative.

```
fprime[x_, y_] := (2 x^2 - 3 y) / (3 x - 2 y^2)
```

```
fprime[1.89, -3.25]
```

```
-1.09312
```

```
fprime[1.89, .87]
```

```
1.09095
```

```
fprime[1.89, 2.38]
```

```
-0.000742207
```

Since the last answer is 0 rounded to two decimals, it's safe say that we have a horizontal tangent line at the point (1.89, 2.38). We can also verify this by looking at the graph. Since the derivative came out as nice numbers for each point, we have checked that the bottom of the derivative is not zero at any of them.

Problem 2

Use the above analysis to find the points at which the equation in **Problem 1b** has horizontal tangent lines. Remember that you should already have defined the **fprime** for it, so just copy, paste, and re-enter it here. In a text cell, explain what you did to solve this problem.

Problem 3

Find the points at which the equation in **Problem 1a** has vertical tangent lines. Look at how you got the horizontal ones in **Problem 2** and see what you can do. You should be able to do most of this problem with paper and pencil. However, in a text cell, explain what you did to solve this problem.

Calculus I Lab 5
Infinity and Inflection

In this Lab, you will study limits to infinity and higher ordered derivatives. You will use a lot of what you learned in the previous Labs. Enjoy.

Problem 1

Let f(x) = $\sqrt{x^2 + \sin x} - x$.

The syntax for limits to infinity is

```
Limit[f[x], x → Infinity]
Limit[f[x], x → -Infinity]
```

a. Compute $\lim_{x\to\pm\infty} f(x)$ in *Mathematica*. Did the one to positive infinity work? Create a table of decimal values so that you can make a guess for the limit good to three decimals. Recall that when you create a table, you have the syntax like {x,10,20,1}, which means that x goes from 10 to 20 in steps of 1. Create a table whose x-values run through the first five or so powers of 10. Hint: If x is a power of 10, then $x = 10^n$ for some integer n. Verify your guess by graphing the function. Also, use a graph to verify *Mathe-*
matica's answer for the limit to negative infinity.

b. Your guess for the limit to positive infinity should be good enough to answer this next question: Find four points at which the graph crosses its horizontal asymptote.

Problem 2

The syntax for computing the nth derivative of f(x) with respect to x is

```
D[f[x], {x, n}]
```

A shortcut for computing the second derivative is **f''[x]**.

a. Compute the first, second, and third derivatives for $f(x) = \frac{x^3 - x^2 + 2}{\sin 2x}$ and $g(t) = \sqrt{t}\ \tan(t^3)$.

b. Plot the graph of $f(x) = \frac{x+1}{x^4+1}$ on the interval [-4,4]. How many inflection points does f appear to have? Now find them ALL and print out the graph with them labeled, by hand. Did you see all of the inflection points?

Calculus 1 Lab 6
Optimization

In this Lab you will study the classic "where do you land" problem. First you will work two specific problems and then you will tackle the more general case. You will have to do a lot of work with pencil and paper before using *Mathematica*.

Problem 1

You are on one side a straight river, which is 200 feet wide. You need to get to a point on the other side that is 500 feet down stream from where you are now. You figure that you can swim at 2 mph and jog at 5 mph. Where on the other side do you land so as to minimize your total travel time, in minutes? Assume that the current of the river is negligible. You must first set up the time function on paper and then use *Mathematica* to find the minimum. What is the domain of your time function?

Problem 2

You want to build a power line from one side a straight river, which is 400 feet wide, to a point on the other side, which is 1000 feet downstream from where you are now. The cost to build over water is $200/ft and over land is $50/ft. At what point on the other side do you land so as to minimize the total cost of the power line? You must first set up the cost function on paper and then use *Mathematica* to find the minimum. What is the domain of your cost function?

Problem 3

Let's see if we can solve this problem more generally. I hope you noticed that your pictures and functions for time and cost were quite similar. Let's use the language of **Problem 2**. Assume that the river is r feet wide and the point on the other side is s feet downstream. For simplicity, assume that the cost to build over land is $1 per foot and to build over water is t dollars per foot, $t>1$. Find the point at which we land so as to minimize the total cost. Be careful in defining your function in *Mathematica* as it will think that r and s are variables. Does this point depend on how far downstream we are to build or the actual costs to build over water and land? On what does it depend? Explain. What is the domain of your cost function?